

Heavy-Light Mesons in Chiral AdS/QCD

Yizhuang Liu* and Ismail Zahed†

Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794-3800, USA

(Dated: November 21, 2016)

We discuss a minimal holographic model for the description of heavy-light and light mesons with chiral symmetry, defined in a slab of AdS space. The model consists of a pair of chiral Yang-Mills and tachyon fields with specific boundary conditions that break spontaneously chiral symmetry in the infrared. The heavy-light spectrum and decay constants are evaluated explicitly. In the heavy mass limit the model exhibits both heavy-quark and chiral symmetry and allows for the explicit derivation of the one-pion axial couplings to the heavy-light mesons.

PACS numbers: 12.39.Jh, 12.39.Hg, 13.30.Eg

I. INTRODUCTION

There is increase interest in the physics of heavy-light mesons and baryons, with a number of newly reported exotics [1–4]. Heavy-light hadrons are characterized by both heavy-quark symmetry [5] and chiral symmetry owing to their light constituents [6, 7], as empirically reported in [8, 9]. Some of the reported exotics were predicted as molecules sometime ago [10–17]. Non-molecular exotics were also suggested using constituent quark models [18], heavy solitonic baryons [19, 20], instantons [21] and QCD sum rules [22]. The molecules are bound heavy mesons near threshold, while the non-molecules are deeply bound quarkonia.

The holographic approach offers a framework for discussing both the spontaneous breaking of chiral symmetry and confinement, in the double limit of large N_c and large 'tHooft coupling $\lambda = g^2 N_c$. A number of descriptions of heavy-light mesons using holography were suggested, without the strictures of chiral symmetry [23]. Recently, we have suggested a holographic construction that exhibits both chiral and heavy quark symmetry [24]. The model is a variant of the Sakai and Sugimoto model [25] with an additional heavy D-brane. The heavy-light mesons are identified with the string low energy modes, and approximated by bi-fundamental and local vector fields in the vicinity of the light probe branes. The chiral pseudo-scalars, vectors and axial-vectors are excitations of the light probe branes with hidden chiral symmetry [26].

The purpose of this paper is to provide an alternative description of the heavy-light mesons and their chiral interactions using a minimal bottom-up approach, whereby left and right flavor gauge fields and flavor tachyons are embedded in a slice of AdS with pertinent boundary conditions. The construction captures the essentials of the holographic principle [27] without the difficulties associated to the D-brane set up. Of course, it lacks the strictures of a first principle approach through D-branes.

Similar approaches for the separate analysis of the light and heavy meson sectors can be found in [28–30].

The organization of the paper is as follows: in section 2 we briefly outline the model and identify the light and heavy fields. In section 3, we detail the analysis of the heavy-light (HL) meson spectrum. In section 4, we derive the axial-vector and vector polarization functions and identify the HL decay constants in closed form. In section 5 we discuss the one-pion interaction to the HL mesons and derive the pertinent axial couplings. Our conclusions are in section 6.

II. ADS/QCD

The holographic construction presented in [24] is based on the top-down approach using non-coincidental $N_f - 1$ light D-branes plus one heavy D-brane, with the HL stringy excitations approximated by bi-fundamental vector fields in the vicinity of the world-volume of the light branes. In the bottom-up approach to follow, we will bypass the details related to the D-brane set up by identifying the pertinent bulk fields in an AdS slab geometry supplemented by appropriate boundary conditions.

A. Model

Consider an AdS geometry in a slab $0 < z \leq z_0$, with a pair of $N_f \times N_f$ vector fields $\mathbf{A}_{L,R}$ and dimensionless tachyon fields $\mathbf{X}_{L,R}$ described by the non-anomalous action

$$\begin{aligned} \mathbf{S}_D = \int d^4 x dz \quad & \left[\frac{1}{4g_5^2} \left(\frac{1}{z} \text{Tr} (\mathbf{F}_L^{MN} \mathbf{F}_{L,MN}) \right) \right. \\ & - \frac{1}{z^3} \text{Tr} |D\mathbf{X}_L|^2 + \frac{3}{z^5} \text{Tr} |\mathbf{X}_L|^2 \\ & + \frac{1}{4g_5^2} \left(\frac{1}{z} \text{Tr} (\mathbf{F}_R^{MN} \mathbf{F}_{R,MN}) \right) \\ & \left. - \frac{1}{z^3} \text{Tr} |D\mathbf{X}_R|^2 + \frac{3}{z^5} \text{Tr} |\mathbf{X}_R|^2 \right] \end{aligned} \quad (1)$$

*Electronic address: yizhuang.liu@stonybrook.edu

†Electronic address: ismail.zahed@stonybrook.edu

with $DX = dX + [\mathbf{A}, \mathbf{X}]$ and $\mathbf{F} = d\mathbf{A} + \mathbf{A}^2$. The coupling $g_5^2 \equiv 6\pi^2/N_c$ is fixed by standard arguments [28, 29] (see below). The anomalous or Chern-Simons (CS) action is

$$S_{\text{CS}} = \frac{N_c}{48\pi^2} \int \text{Tr} \left(\mathbf{A}_L \mathbf{F}_L^2 - \frac{1}{2} \mathbf{A}_L^2 \mathbf{F}_L + \frac{1}{10} \mathbf{A}_L^5 \right) + \frac{N_c}{48\pi^2} \int \text{Tr} \left(\mathbf{A}_R \mathbf{F}_R^2 - \frac{1}{2} \mathbf{A}_R^2 \mathbf{F}_R + \frac{1}{10} \mathbf{A}_R^5 \right) \quad (2)$$

with the integration carried over a slice of AdS with no surface terms added. The matrix valued 1-form gauge field is

$$\mathbf{A} = \begin{pmatrix} A & \Phi \\ -\Phi^\dagger & 0 \end{pmatrix} \quad (3)$$

The effective fields in the field strengths are (M, N) run over (μ, z)

$$\mathbf{F}_{MN} = \begin{pmatrix} F_{MN} - \Phi_{[M} \Phi_{N]}^\dagger & \partial_{[M} \Phi_{N]} + A_{[M} \Phi_{N]} \\ -\partial_{[M} \Phi_{N]}^\dagger - \Phi_{[M}^\dagger A_{N]} & -\Phi_{[M}^\dagger \Phi_{N]} \end{pmatrix} \quad (4)$$

The light degrees of freedom are described by the vector fields $A_{L,R}$, with the axial and vector assignments defined by their IR boundary condition at $z = z_0$. Specifically, in the infrared at $z = z_0$ we define

$$A_{L,R}(x, z_0) = +\epsilon_{V,A} A_R(x, z_0) \\ A'_{L,R}(x, z_0) = -\epsilon_{V,A} A'_R(x, z_0) \quad (5)$$

with $\epsilon_V = +1$ for vector fields and $\epsilon_A = -1$ for axial-vector fields. In the ultraviolet we identify $A_{L,R}(z = 0) = \mathbb{J}_{L,R}$ with their boundary sources. For the pion field, we note the extra rigid flavor gauge symmetry at the infrared boundary

$$A_R(x, z_0) \rightarrow g_- A_R(x, z_0) g_-^{-1} \\ A_L(x, z_0) \rightarrow g_+ A_L(x, z_0) g_+^{-1} \quad (6)$$

The pion field is identified with the double holonomies

$$U(x) = e^{\frac{i}{f_\pi} \pi(x)} \equiv P e^{-\int_{z_0}^0 A_{Lz}(x, z') dz'} P e^{-\int_0^{z_0} A_{Rz}(x, z') dz'} \quad (7)$$

with the squared pion decay constant $f_\pi^2 = 2/g_5^2 z_0^2$ [29].

The heavy degrees of freedom are described by the vector field Φ in (3). They acquire a mass through their coupling to the background tachyon fields $\mathbf{X}_{L,R}$,

$$\mathbf{X}_L = \mathbf{X}_R \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & X(z) \end{pmatrix} \quad (8)$$

From (1), the linearized equation for $X(z)$ reads

$$\frac{d}{dz} \left(\frac{1}{z^3} \frac{dX}{dz} \right) + \frac{3}{z^5} X = 0 \quad (9)$$

which is solved by

$$X(z) \approx c_1 z + c_2 z^3 \quad (10)$$

The constants in (10) are fixed by the holographic dictionary [27, 28] near the UV boundary ($z \approx 0$)

$$X(z) \approx Mz + \langle \bar{Q}Q \rangle z^3 \quad (11)$$

In the heavy quark limit $\langle \bar{Q}Q \rangle \rightarrow 0$, so $X(z) \approx Mz$

III. HEAVY-LIGHT SPECTRUM

When restricted to only the HL vector degrees of freedom, the field-strength 2-forms in (4) are equal

$$(\mathbf{F}_{L,R})_{MN} \rightarrow \begin{pmatrix} -\Phi_{[M} \Phi_{N]}^\dagger & \partial_{[M} \Phi_{N]} \\ -\partial_{[M} \Phi_{N]}^\dagger & -\Phi_{[M}^\dagger \Phi_{N]} \end{pmatrix} \quad (12)$$

Inserting (12) into (1) yields to quadratic order in Φ

$$S_\Phi = -\frac{1}{2g_5^2} \int dz d^4x \frac{1}{z} \\ \times [(\partial_\mu \Phi_\nu^\dagger - \partial_\nu \Phi_\mu^\dagger)(\partial^\mu \Phi^\nu - \partial^\nu \Phi^\mu) \\ + (\partial_\mu \Phi^\dagger - \partial_z \Phi_\mu^\dagger)(\partial^\mu \Phi - \partial^z \Phi^\mu)] \quad (13)$$

Now, we consider the spectrum of the heavy-light mesons. For that, we need the off-diagonal fluctuations of the tachyonic field as they mix with the longitudinal vector modes

$$\mathbf{X}_{L,R} \rightarrow \begin{pmatrix} X_1 & X_2 \\ X_2^\dagger & Mz \end{pmatrix} \quad (14)$$

Since the equations of motion for the L, R are the same, we will omit these labels unless specified otherwise. The general equations of motion can be obtained from (1) as

$$z \partial_M \frac{1}{z} F^{MN} \\ - 2g_5^2 (M^2 \Phi^N + \frac{M}{z} \partial^N X_2 - \frac{M}{z^2} \delta_{N,z} X_2) = 0 \\ \partial^M \frac{1}{z^3} \partial_M X_2 + \frac{3}{z^5} X_2 + M \left(\frac{\Phi_z}{z^3} + \partial^M \frac{1}{z^2} \Phi_M \right) = 0 \quad (15)$$

A. Transverse modes

The equations of motion for the transverse modes with $\partial^\mu \Phi_\mu = 0$ and $X_2 = \Phi_z = 0$, follow through the substitution $\Phi_\mu(p, z) = \phi_n(p, z)\epsilon_\mu(p)$ in (15). These modes decouple from the tachyonic modes and satisfy

$$\frac{d}{dz} \left(\frac{1}{z} \frac{d\phi_n}{dz} \right) + \frac{1}{z} k^2(p) \phi_n = 0 \quad (16)$$

with

$$\begin{aligned} k^2(p) &= -p^2 - m_Q^2 \\ m_Q^2 &= 2g_5^2 M^2 \end{aligned} \quad (17)$$

where we have identified m_Q as the (bare) heavy quark mass. (16) is solved in terms of Bessel functions

$$\phi(p, z) = C_1 z J_1(k(p)z) + C_2 z Y_1(k(p)z) \quad (18)$$

The transverse modes satisfy the mass shell-condition $p^2 = -m_n^2$ with the (unrenormalized) eigenmodes and eigenvalues

$$\phi_n(z) = z J_1(k_n z), \quad m_n^2 = k_n^2 + m_Q^2 \quad (19)$$

Here the k_n are fixed by the IR boundary conditions (5),

$$\begin{aligned} J_0(k_{2n} z_0) &= 0 & \text{vector} \\ J_1(k_{2n+1} z_0) &= 0 & \text{axial} \end{aligned} \quad (20)$$

For the lowest states, we have explicitly $k_0 = 2.40/z_0$ (vector), $k_1 = 3.83/z_0$ (axial). The HL meson wavefunctions (19-20) are independent of the heavy quark mass m_Q in contrast to those developed in the HL holographic variant of the Sakai-Sugimoto model in [24]. The reason is that in (16) the heavy quark mass m_Q appears always in the combination $k(p)$ which is kinematical. This is not the case in [24] where m_Q^2 is warped differently than p^2 .

The splitting between the axial-vector states (n-odd) and the vector states (n-even) vanishes in the heavy quark limit. Indeed, for the two lowest states

$$\begin{aligned} \Delta_Q &= (m_Q^2 + k_1^2)^{\frac{1}{2}} - (m_Q^2 + k_0^2)^{\frac{1}{2}} \\ &\approx \frac{k_1^2 - k_0^2}{2m_Q} = \frac{8.91}{2m_Q z_0^2} \end{aligned} \quad (21)$$

Assuming that the confining wall position z_0 is universal, (21) implies the splitting ratio $\Delta_C/\Delta_B \approx m_B/m_C \approx 3.28$ for charm to bottom HL mesons, which is larger than the empirical ratio $\Delta_C/\Delta_B = 420/396 = 1.06$ [9]. We note that our derivation of the spectrum (19) was carried with a zero light quark condensate, $\langle \bar{q}q \rangle = 0$. This can be remedied by allowing for a background $X_1(z)$ in (8).

B. Longitudinal modes

The longitudinal part of Φ_μ mixes with the tachyonic mode X_2 . Indeed, the tachyonic kinetic contribution in (1) amounts to several contributions

$$\begin{aligned} |D\mathbf{X}|^2 &= |\partial_M X_1 + A_M X_1 - X_1 A_M + \Phi_M X_2^\dagger + X_2 \Phi_M^\dagger|^2 \\ &+ 2|\partial_M X_2 + A_M X_2 + (X(z) - X_1)\Phi_M^\dagger|^2 \\ &+ (\partial_z X(z) - (\Phi_z^\dagger X_2 + X_2^\dagger \Phi_z))^2 \end{aligned} \quad (22)$$

with explicit $X\Phi$ mixing terms. Inserting (22) into (1) and keeping only the $X\Phi$ contributions give

$$\begin{aligned} \mathcal{L}_{X\Phi} &= -\frac{1}{2g_5^2 z} \partial_{[M} \Phi_{N]}^\dagger \partial^{[M} \Phi^{N]} + \frac{6}{z^5} X_2^\dagger X_2 - \frac{2}{z^3} \partial^M X_2^\dagger \partial_M X_2 \\ &- \frac{2}{z} M^2 \Phi^\dagger M \Phi + \frac{2M}{z^3} (\Phi_z^\dagger X_2 + X_2^\dagger \Phi_z) \\ &- \frac{2M}{z^2} (\partial^M X_2^\dagger \Phi_M + \Phi^{M\dagger} \partial_M X_2) \end{aligned} \quad (23)$$

Using the longitudinal mode decompositions

$$\begin{aligned} \Phi_\mu(p, z) &= \partial_\mu(\phi(p, z)e^{ipx}) \\ \Phi_z(p, z) &= \Phi_1(p, z)e^{ipx} \\ X_2(p, z) &= z^2 M \Phi_2(p, p) \end{aligned} \quad (24)$$

in (23) we have

$$\begin{aligned} \mathcal{L}_{X\Phi} &= -\frac{p^2}{g_5^2 z} \left| \Phi_1 - \frac{d\phi}{dz} \right|^2 - \frac{2}{z^3} \left| \frac{dX_2}{dz} \right|^2 - \frac{2p^2}{z^3} |X_2|^2 \\ &+ \frac{6}{z^5} |X_2|^2 - \frac{2M^2}{z} |\Phi_1|^2 - \frac{2M^2 p^2}{z} |\phi|^2 \\ &+ \frac{2M}{z^3} (X_2^\dagger \Phi_1 + \Phi_1^\dagger X_2) - \frac{2Mp^2}{z^2} (X_2^\dagger \phi + \phi^\dagger X_2) \\ &- \frac{2M}{z^2} \left(\frac{dX_2^\dagger}{dz} \Phi_1 + \Phi_1^\dagger \frac{dX_2}{dz} \right) \end{aligned} \quad (25)$$

which shows that Φ_1 is a constraint field following from the gauge symmetry that causes the longitudinal field ϕ and the tachyon field X_2 to mix. Varying with respect to $\Phi_{1,2}$ and ϕ , yield the coupled equations

$$\begin{aligned} -p^2 \left(\Phi_1 - \frac{d\phi}{dz} \right) - 2g_5^2 M^2 \left(\Phi_1 + \Phi_2 + z \frac{d\Phi_2}{dz} \right) &= 0 \\ + z \frac{d}{dz} \frac{1}{z} \left(\frac{d\phi}{dz} - \Phi_1 \right) - 2g_5^2 M^2 (\phi + z\Phi_2) &= 0 \\ -p^2 \left(\Phi_2 + \frac{\phi}{z} \right) + \frac{d^2 \Phi_2}{dz^2} + \frac{1}{z} \frac{d(\Phi_1 + \Phi_2)}{dz} - \frac{\Phi_1 + \Phi_2}{z^2} &= 0 \end{aligned} \quad (26)$$

The constraint is readily unwound in terms of the longitudinal modes

$$\Phi_1 = \frac{p^2}{p^2 + m_Q^2} \frac{d\phi}{dz} = -\frac{p^2}{k^2(p)} \frac{d\phi}{dz} \quad (27)$$

Inserting (27) in (26), shows that there is only one independent combination $\tilde{\phi} = \phi + z\Phi_2$ satisfying

$$\frac{d^2\tilde{\phi}}{dz^2} - \frac{1}{z} \frac{d\tilde{\phi}}{dz} + k^2(p)\tilde{\phi} = 0 \quad (28)$$

The massive longitudinal modes in (28) obey the same equation as the massive transverse modes in (16). The redundancy of the degrees of freedom in (26) allows the gauge choice $\Phi_2 = 0$ for instance, to represent the longitudinal modes in (28). The explicit solutions are

$$\tilde{\phi}(p, z) = c_1 z J_1(k(p)z) + c_2 z Y_1(k(p)z) \quad (29)$$

Only the modes $zJ_1(kz)$ are square integrable near the boundary. We identify the pseudo-scalar HL modes by enforcing $\tilde{\phi}(p, z_0) = 0$, and the scalar HL modes by enforcing $\tilde{\phi}'(p, z_0) = 0$ at the wall.

C. Canonical HL actions

To show how the canonical action for the massive HL scalars and pseudo-scalars emerge from (1) in light of our identification above, consider the explicit mode decomposition for the longitudinal fields in the gauge with $\Phi_2 = 0$,

$$\begin{aligned} \Phi_z^L(x, z) &= \sum_n \frac{-p_n^2}{k^2(p_n)} \frac{d\phi_n}{dz} D_n(x) \\ \Phi_\mu^L(x, z) &= \sum_n \phi_n(z) \partial_\mu D_n(x) \end{aligned} \quad (30)$$

Inserting (30) into (1) and keeping only the quadratic contributions in D_n , yield

$$\begin{aligned} \mathbf{S}_D &= +2 \int d^4x \sum_{m,n} \partial^\mu D_m^\dagger \partial_\mu D_n \int dz \frac{m_Q^2 p_n^2}{k^2(p_n)} \frac{\phi_n \phi_m}{g_5^2 z} \\ &\quad -2 \int d^4x \sum_{mn} m_Q^2 D_m^\dagger D_n \int dz \frac{p_n^2 p_m^2}{k^2(p_n)} \frac{\phi_m \phi_n}{g_5^2 z} \end{aligned} \quad (31)$$

(31) suggests that we normalize the eigenmodes in (30) using

$$\int_0^{z_0} dz \frac{m_Q^2 p_n^2}{k^2(p_n)} \frac{\phi_n \phi_m}{g_5^2 z} = -\frac{\delta_{mn}}{2} \quad (32)$$

which also supports the identity

$$\int \frac{f_m f_n}{g_5^2 z} dz = \frac{-p_n^2}{m_Q^2} \delta_{mn} = \left(1 + \frac{k_n^2}{m_Q^2}\right) \delta_{mn} \quad (33)$$

for the derivative modes

$$f_n(z) = -\frac{p_n^2}{k^2(p_n)} \frac{d\phi_n}{dz} \quad (34)$$

In the heavy quark limit, (32) brings (31) to the canonical action form for the HL scalars and pseudo-scalars,

$$\mathbf{S}_D = - \int d^4x \sum_n (|\partial_\mu D_n|^2 + m_n^2 |D_n|^2) \quad (35)$$

Similar arguments for the transverse modes with the pertinent normalizations, yield the canonical action for the HL vectors and axial-vectors

$$\mathbf{S}_{D_\mu} = - \int d^4x \sum_n \left(\frac{1}{2} |D_{\mu\nu n}|^2 + m_n^2 |D_{\mu n}|^2 \right) \quad (36)$$

It follows from (35-36) together with the boundary conditions at the wall (5), that the pseudo-scalar and vector spectra (odd-parity) are degenerate, and that the scalar and axial-vector spectra (even-parity) are degenerate for any finite m_Q in the present holographic set up. This degeneracy follows from the rigid $O(4)$ symmetry of the vector fields in (1) in 5-dimensions.

IV. AXIAL-VECTOR AND VECTOR CORRELATORS

The vector and axial polarization functions in walled AdS/QCD can be derived using standard holographic arguments [27–29]. In particular, the bulk interpolating chiral vector fields are

$$\Phi_\mu^{L,R}(p, z) = \frac{\mathcal{V}(p, z)}{\mathcal{V}(p, \epsilon)} \mathbb{J}_\mu^{L,R}(p) \quad (37)$$

with the bulk-to-boundary propagator satisfying the analogue of (28),

$$\frac{d^2\mathcal{V}}{dz^2} - \frac{1}{z} \frac{d\mathcal{V}}{dz} + k^2(p)\mathcal{V} = 0 \quad (38)$$

with similar IR boundary conditions as in (5). The solutions are

$$\begin{aligned}
\mathbb{J}^L &= -\mathbb{J}^R : \text{ axial} \\
\mathcal{V}(p, z) &= zY_1(k(p)z_0)J_1(k(p)z) - zJ_1(k(p)z_0)Y_1(k(p)z) \\
\mathbb{J}^L &= +\mathbb{J}^R : \text{ vector} \\
\mathcal{V}(p, z) &= zY_0(k(p)z_0)J_1(k(p)z) - zJ_0(k(p)z_0)Y_1(k(p)z)
\end{aligned} \tag{39}$$

A. Polarization functions

The corresponding boundary action for the HL mesons in walled AdS/QCD is now readily constructed using standard arguments [27–29], with the result

$$\mathbf{S}_B[\mathbb{V}] = + \int \frac{d^4 q}{(2\pi)^4} \mathbb{V}_\mu^\dagger(q) \left(\frac{1}{2g_5^2 \epsilon} \frac{\partial_z \mathcal{V}(q, \epsilon)}{\mathcal{V}(q, \epsilon)} \right) \mathbb{V}^\mu(q) \tag{40}$$

and similarly for $\mathbf{S}_B[\mathbb{A}]$. Here, the sources are

$$\begin{aligned}
\mathbb{V}_\mu &= \mathbb{J}_\mu^L + \mathbb{J}_\mu^R \\
\mathbb{A}_\mu &= \mathbb{J}_\mu^L - \mathbb{J}_\mu^R
\end{aligned} \tag{41}$$

The vector polarization function is obtained by inserting (39) (second relation) in (40),

$$\begin{aligned}
\Pi_V(q) &= \\
&= -\frac{1}{2g_5^2} \frac{k(q)}{\epsilon} \frac{Y_0(k(q)z_0)J_0(k(q)\epsilon) - J_0(k(q)z_0)Y_0(k(q)\epsilon)}{Y_0(k(q)z_0)J_1(k(q)\epsilon) - J_0(k(q)z_0)Y_1(k(q)\epsilon)}
\end{aligned} \tag{42}$$

Using the short distance part of the Neumann function $Y_1(x) \approx -2/\pi x$ as $\epsilon \rightarrow 0$, we can reduce (42) to

$$\Pi_V(q) = -\frac{\pi k^2(q)}{4g_5^2} \frac{Y_0(k(q)z_0)}{J_0(k(q)z_0)} + \frac{1}{2g_5^2} k^2(q) \ln(k(q)\epsilon) \tag{43}$$

The first contribution in (43) displays a string of poles that reproduces the vector spectrum in (19). The last contribution reduces to the free HL correlator as $k^2(q) \approx q^2 \rightarrow \infty$, provided that we identify $g_5^2 = 6\pi^2/N_c$ [28, 29].

Similarly, the axial polarization function is obtained by inserting (39) (first relation) in (40). The result is

$$\begin{aligned}
\Pi_A(q) &= \\
&= -\frac{1}{2g_5^2} \frac{k(q)}{\epsilon} \frac{Y_1(k(q)z_0)J_0(k(q)\epsilon) - J_1(k(q)z_0)Y_0(k(q)\epsilon)}{Y_1(k(q)z_0)J_1(k(q)\epsilon) - J_1(k(q)z_0)Y_1(k(q)\epsilon)}
\end{aligned} \tag{44}$$

which can be reduced to

$$\Pi_A(q) = -\frac{\pi k^2(q)}{4g_5^2} \frac{Y_1(k(q)z_0)}{J_1(k(q)z_0)} + \frac{1}{2g_5^2} k^2(q) \ln(k(q)\epsilon) \tag{45}$$

as $\epsilon \rightarrow 0$. The poles of (45) reproduce the axial-vector spectrum in (19). The free contribution in (45) is identical to that in (43) as it should.

B. Decay constants

The residues at the poles of the polarization functions (42) and (44) correspond to the HL vector f_{V_n} and axial-vector f_{A_n} decay constants respectively. For that, we note that at the poles, (42) satisfies the identity [31]

$$\begin{aligned}
&\frac{Y_0(k(q)z_0)J_0(k(q)\epsilon) - J_0(k(q)z_0)Y_0(k(q)\epsilon)}{J_0(k(q)z_0)} = \\
&= -\frac{4}{\pi} \sum_n \frac{1}{(z_0 J_1(\kappa_{2n}))^2} \frac{1}{k^2(q) - k_{2n}^2}
\end{aligned} \tag{46}$$

with $\kappa_{2n} = k_{2n}z_0$ the zeros of $J_0(\kappa_{2n}) = 0$ in (20). Inserting (46) in (42) and recalling that $k^2(q) = -q^2 - m_Q^2$, and that $m_n^2 = m_Q^2 + k_n^2$, we obtain

$$\Pi_V(q) = \sum_n \left(\frac{k_{2n}/m_{2n}}{g_5 z_0 J_1(\kappa_{2n})} \right)^2 \frac{m_{2n}^2}{q^2 + m_{2n}^2} \tag{47}$$

with the vector decay constants

$$f_{V_n} = \frac{k_{2n}/m_{2n}}{g_5 z_0 J_1(\kappa_{2n})} \tag{48}$$

Similar arguments for the axial correlator (44) give

$$\Pi_A(q) = \sum_n \left(\frac{k_{2n+1}/m_{2n+1}}{g_5 z_0 J_0(\kappa_{2n+1})} \right)^2 \frac{m_{2n+1}^2}{q^2 + m_{2n+1}^2} \tag{49}$$

with $\kappa_{2n+1} = k_{2n+1}z_0$ the zeros of $J_1(\kappa_{2n+1}) = 0$ in (20). The axial-vector decay constant commonly referred to as the pseudo-scalar decay constants follows from (49)

$$f_{A_n} = \frac{k_{2n+1}/m_{2n+1}}{g_5 z_0 J_1(\kappa_{2n+1})} \tag{50}$$

Using the pion decay constant as defined in (7), and the Bessel asymptotics for large arguments [31]

$$J_n(x) \approx \left(\frac{2}{\pi x} \right)^{\frac{1}{2}} \cos \left(x - (2n+1) \frac{\pi}{4} \right) \tag{51}$$

we can recast the decay constants as the dimensionless ratios

$$\left(\frac{f_{V_n}}{f_\pi}, \frac{f_{A_n}}{f_\pi}\right) = \frac{\pi}{2\sqrt{2}} \left(\frac{k_{2n}}{m_{2n}}, \frac{k_{2n+1}}{m_{2n+1}}\right) \quad (52)$$

In particular, we find that the ratio of the B-meson f_B to D-meson f_D decay constant is

$$\frac{f_B}{f_D} = \frac{m_D}{m_B} = \frac{1869}{5279} = 0.35 \quad (53)$$

which is smaller than the lattice reported ratio $f_B/f_D = 0.88$ [32]. We recall that general arguments suggest $f_B/f_D = (m_D/m_B)^{\frac{1}{2}} = 0.55$ [33].

V. CHIRAL AXIAL COUPLINGS

Since our set up is chirally symmetric, with the wall boundary conditions (5) breaking the symmetry spontaneously as in [25], we can also address the pion interactions with the HL mesons in the AdS slice, with the pion field identified as in (7). In particular, the zero mode contribution to $A_{L,R}$ is

$$A_{L,R}(x, z) \approx \frac{i\pi(x)}{f_\pi} \psi'_0(z) \quad (54)$$

with the chiral pion zero mode now identified as

$$\psi_0(z) = \frac{1}{2} \left(1 - \frac{z^2}{z_0^2}\right) \quad (55)$$

The chiral effective action with HL light quarks for walled AdS/QCD follows the same arguments as those developed in [24].

In the presence of the pion field, the HL modes get dressed by a pion field to enforce the correct chiral transformations as detailed in [24]. In the one-pion approximation, the dressed and transverse boundary mode decomposition in the chiral effective action reads

$$\begin{aligned} \Phi_\mu^T(x, z) &\approx \left(1 - \frac{i}{f_\pi} \psi_0(z) \pi(x)\right) \sum_n \phi_n(z) D_{n\mu}(x) \\ \Phi_z^T(x, z) &= 0 \end{aligned} \quad (56)$$

while for the dressed and longitudinal mode decomposition we have

$$\begin{aligned} \Phi_\mu^L(x, z) &\approx \left(1 - \frac{i}{f_\pi} \psi_0(z) \pi(x)\right) \\ &\quad \times \sum_n \frac{k_n \phi_n(z)}{m_Q m_n} \partial_\mu D_n \\ \Phi_z^L(x, z) &\approx \left(1 - \frac{i}{f_\pi} \psi_0(z) \pi(x)\right) \\ &\quad \times \sum_n \frac{m_n}{m_Q k_n} \frac{d\phi_n}{dz} D_n(x) \end{aligned} \quad (57)$$

We note the sign change in our definition of the pion field in comparison to the one used in [24], owing to the opposite z-direction for the IR and UV boundaries between the two analyses.

In the heavy quark limit $m_Q \rightarrow \infty$, Φ_μ^L is subleading and will be disregarded. The modes common to both Φ_μ^T and Φ_z^L share now the same normalizations

$$\phi_n = c_n z J_1(k_n z) \quad \text{and} \quad 2 \int_0^{z_0} dz \frac{\phi_n^2}{g_5^2 z} = 1 \quad (58)$$

with the c_n fixed by

$$\begin{aligned} c_{2n}^2 z_0^2 J_1^2(k_{2n} z_0) &= g_5^2 & \text{vector} \\ c_{2n+1}^2 z_0^2 J_0^2(k_{2n+1} z_0) &= g_5^2 & \text{axial} \end{aligned} \quad (59)$$

The one-pion interaction terms with the HL mesons follow the same analysis as that in [24] with two minor changes: 1/ the substitution of the warping factors in the DBI action in [24] by the corresponding warpings in walled AdS; 2/ the substitution of the HL and pion mode in [24] by their corresponding modes in walled AdS, i.e.

$$\begin{aligned} \phi_n(z) &\rightarrow \phi_n(z) & \text{transverse} \\ \tilde{\phi}_n(z) &\rightarrow \frac{1}{k_n} \frac{d\phi_n}{dz} & \text{longitudinal} \end{aligned} \quad (60)$$

A. $g_{H,G}$ couplings

The one-pion interaction to the $(H, G) = (0^\mp, 1^\mp)$ multiplets defined in standard non-relativistic form

$$\begin{aligned} H \rightarrow \mathbb{H}_+ &= \frac{e^{-iMx_0}}{\sqrt{2M}} (i\gamma_5 D + \gamma_\mu D^\mu) \frac{1+\gamma_0}{2} \\ G \rightarrow \mathbb{G}_+ &= \frac{e^{-iMx_0}}{\sqrt{2M}} (D_0 + \gamma_\mu \gamma_5 D_1^\mu) \frac{1+\gamma_0}{2} \end{aligned} \quad (61)$$

follows from the CS contributions in (2) by using the one-pion expanded forms (56-57) and retaining only the positive energy contributions (61) as in [24]. This amounts to a special deformation of the CS contribution in the HL sector as detailed in [24]. The result for the one-pion coupling to the odd-parity H-multiplet is

$$\begin{aligned} S_{\text{CS}}^+ &= -\frac{iN_c}{32\pi^2 f_\pi} \int dz \psi'_0(z) \phi_0^2 \\ &\quad \times \int d^4x \text{Tr} \partial_i \pi (D_i D^\dagger - D D_i^\dagger + \epsilon^{ijk} D_k D_j^\dagger) \end{aligned} \quad (62)$$

from which we read the axial coupling

$$\begin{aligned}
g_H &= -\frac{N_c}{16\pi^2} \int_0^{z_0} dz \phi_0^2 \psi'_0(z) \\
&= +\frac{3}{16} \frac{\int_0^1 x^3 J_1^2(k_0 x)}{\int_0^1 x J_1^2(k_0 x)} = 0.10
\end{aligned} \tag{63}$$

The result is smaller than the reported value of $g_H = 0.65$, as measured through the charged pion decay $D^* \rightarrow D + \pi$ [9]. The one-pion coupling to the even-parity G-multiplet follows from (63) through the substitution $\phi_0 \rightarrow \phi_1$ ($k_0 \rightarrow k_1$)

$$\begin{aligned}
g_G &= -\frac{N_c}{16\pi^2} \int_0^{z_0} dz \phi_1^2 \psi'_0(z) \\
&= +\frac{3}{16} \frac{\int_0^1 x^3 J_1^2(k_1 x)}{\int_0^1 x J_1^2(k_1 x)} = 0.14
\end{aligned} \tag{64}$$

Both results are to be compared to $g_H = g_G = 27/4\lambda \approx 3/4$ for $\lambda \approx 9$ in the top-down approach developed in [24].

B. g_{HG} coupling

Similarly, the one-pion cross-multiplet coupling g_{HG} follows from the expansion of the bulk contributions in (1) after the insertions of (56-57) for the H-multiplet (odd-parity), and similarly for the G-multiplet (even-parity) with the additional substitution $(D, D^\mu) \rightarrow (D_0, D_1^\mu)$. The result for the one-pion cross-multiplet coupling term is

$$\begin{aligned}
S_{HG}^\pi &= -\frac{4\kappa}{f_\pi} \int f \phi_0 \phi_1 \psi_0(z) dz \int d^4 x \text{Tr} \partial_0 \pi (D_i D_{1i}^\dagger + D_{1i} D_i^\dagger) \\
&\quad -\frac{2\kappa}{f_\pi} \int g \tilde{\phi}_0 \tilde{\phi}_1 \psi_0(z) dz \int d^4 x \text{Tr} \partial_0 \pi (D D_0^\dagger + D_0 D^\dagger)
\end{aligned} \tag{65}$$

with $2\kappa = 1/g_5^2$ and $2f = g = 1/z$ in agreement with the result in [24]. Using (60), we have

$$\begin{aligned}
S_{HG}^\pi &= -\frac{1}{f_\pi} \int_0^{z_0} \frac{2}{g_5^2 z} \psi_0 \phi_0 \phi_1 \\
&\quad \times \int d^4 x \text{Tr} \partial_0 \pi (D_{1i} D_i^\dagger + \text{c.c.}) \\
&\quad -\frac{1}{f_\pi} \int_0^{z_0} dz \frac{2}{g_5^2 z} \psi_0 \frac{1}{k_0 k_1} \frac{d\phi_0}{dz} \frac{d\phi_1}{dz} \\
&\quad \times \int d^4 x \text{Tr} \partial_0 \pi (D_0 D^\dagger + \text{c.c.})
\end{aligned} \tag{66}$$

We now note the identity

$$\begin{aligned}
&\int_0^{z_0} \psi_0 \frac{1}{k_0 k_1 z} \phi'_0 \phi'_1 = \\
&= -\int_0^{z_0} \frac{\psi_0}{k_0 k_1} \phi_0 \frac{d}{dz} \frac{1}{z} \frac{d\phi_1}{dz} + \frac{1}{2} \int_0^{z_0} \frac{1}{k_0 k_1 z_0^2} \phi_0 \frac{d\phi_1}{dz}
\end{aligned} \tag{67}$$

Interchanging the labels 0, 1 in the integrand, and noticing that one of the (ϕ_0, ϕ_1) mode vanishes at z_0 , we obtain

$$\int_0^{z_0} \psi_0 \frac{1}{k_0 k_1 z} \phi'_0 \phi'_1 = \frac{1}{2} \left(\frac{k_0}{k_1} + \frac{k_1}{k_0} \right) \int_0^{z_0} dz \frac{\psi_0}{z} \phi_0 \phi_1 \tag{68}$$

Inserting (68) into (66) yields

$$\begin{aligned}
S_{HG}^\pi &= -\frac{1}{f_\pi} \int_0^{z_0} \frac{2}{g_5^2 z} \psi_0 \phi_0 \phi_1 \\
&\quad \times \left(\int d^4 x \text{Tr} \partial_0 \pi (D_{1i} D_i^\dagger + \text{c.c.}) \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{k_0}{k_1} + \frac{k_1}{k_0} \right) \right. \\
&\quad \left. \times \int d^4 x \text{Tr} \partial_0 \pi (D_0 D^\dagger + \text{c.c.}) \right)
\end{aligned} \tag{69}$$

with

$$\frac{1}{2} \left(\frac{k_0}{k_1} + \frac{k_1}{k_0} \right) = \frac{1}{2} \left(\frac{2.4}{3.83} + \frac{3.83}{2.4} \right) = 1.11 \tag{70}$$

which is about 1 in (69). The small deviation from 1 maybe traced to the fact that the longitudinal modes Φ_z^L in (57) may still develop a nontrivial mixing with the X_2 tachyon mode at the one-pion interaction level requiring a further constraint to bring it to 1. This notwithstanding, the second contribution in (69) matches the first contribution with heavy quark symmetry manifest. With this in mind, the cross-multiplet coupling in the heavy mass limit, can be read from the pre-factor in (67)

$$g_{HG} = -\int_0^{z_0} \frac{2}{g_5^2 z} \psi_0 \phi_0 \phi_1 = -0.45 \tag{71}$$

This result is to be compared to $g_{HG} \approx 0.18$ for charmed mesons and $g_{HG} \approx 0.10$ for bottom mesons established in [24]. The origin of the difference lies in the fact that the HL mesonic wavefunctions in walled AdS do not depend on the heavy quark mass as we noted in section IIIA above. (71) yields to larger partial widths for the $G \rightarrow H + \pi$ decays in comparison to those discussed in [24].

VI. CONCLUSIONS

We have presented a minimal bottom-up holographic approach to the HL mesons interacting with the lightest pseudoscalar mesons. The holographic construction

assumes bulk chiral vector fields interacting with tachyonic modes sourced by a mass, in a slice of AdS. The HL vector and axial-vector modes are identified with the transverse modes of the chiral vector fields, while the scalar and pseudo-scalar modes are identified with the longitudinal modes of the chiral vector fields. They are massive through their coupling to the tachyonic modes by the Higgs mechanism, and degenerate because of the underlying $O(4)$ rigid flavor symmetry of the bulk Yang-Mills action in 5-dimensions.

The HL meson spectrum does not reggeizes, a well-known shortcoming of the hard wall model. This can be remedied by using a soft wall for instance [34], with no major changes in our analysis. The splitting between the consecutive vector and axial-vector multiplets vanishes in the heavy quark limit. We have explicitly computed the HL correlation functions using the holographic principle, and extracted the pertinent HL decay constants. The ratio of the B-meson to D-meson decay constants is found to be half the ratio reported in current lattice measurements and experiments.

We have made explicit use of the HL effective action

to extract the pertinent axial charges for the low lying HL multiplets $H, G = (0^\mp, 1^\mp)$ in the heavy quark limit. Holography shows that the axial couplings are about equal with $g_H = 0.10$ and $g_G = 0.14$, but smaller than the reported experimental value of $g_H = 0.65$. The one-pion cross-multiplet coupling is found to be $g_{GH} = -0.45$.

The present walled AdS/QCD model can be improved in many ways, through the use of a soft wall or improved holographic QCD [35] for instance. However, it does provide a simple framework for discussing both chiral and heavy quark symmetry with applications to analyze the electromagnetic decays of HL mesons, as well as the description of HL baryons as holographic solitonic bound states. Some of these issues will be addressed next.

VII. ACKNOWLEDGEMENTS

This work was supported by the U.S. Department of Energy under Contract No. DE-FG-88ER40388.

-
- [1] I. Adachi [Belle Collaboration], arXiv:1105.4583 [hep-ex]; A. Bondar *et al.* [Belle Collaboration], Phys. Rev. Lett. **108**, 122001 (2012) [arXiv:1110.2251 [hep-ex]].
 - [2] M. Ablikim *et al.* [BESIII Collaboration], Phys. Rev. Lett. **110**, 252001 (2013) [arXiv:1303.5949 [hep-ex]].
 - [3] V. M. Abazov *et al.* [D0 Collaboration], [arXiv:1602.07588 [hep-ex]].
 - [4] R. Aaij *et al.* [LHCb Collaboration], arXiv:1606.07895 [hep-ex]; R. Aaij *et al.* [LHCb Collaboration], arXiv:1606.07898 [hep-ex].
 - [5] N. Isgur and M. B. Wise, Phys. Rev. Lett. **66** (1991) 1130; A. V. Manohar and M. B. Wise, “Heavy quark physics,” Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **10**, 1 (2000).
 - [6] M. A. Nowak, M. Rho and I. Zahed, Phys. Rev. D **48**, 4370 (1993) [hep-ph/9209272]; M. A. Nowak, M. Rho and I. Zahed, Acta Phys. Polon. B **35**, 2377 (2004) [hep-ph/0307102].
 - [7] W. A. Bardeen and C. T. Hill, Phys. Rev. D **49** (1994) 409 [hep-ph/9304265]; W. A. Bardeen, E. J. Eichten and C. T. Hill, Phys. Rev. D **68**, 054024 (2003) [hep-ph/0305049].
 - [8] B. Aubert *et al.* [BaBar Collaboration], Phys. Rev. Lett. **90**, 242001 (2003) [hep-ex/0304021].
 - [9] D. Besson *et al.* [CLEO Collaboration], Phys. Rev. D **68**, 032002 (2003) Erratum: [Phys. Rev. D **75**, 119908 (2007)] [hep-ex/0305100]; A. Anastassov *et al.* [CLEO Collaboration], Phys. Rev. D **65**, 032003 (2002) [hep-ex/0108043].
 - [10] M. B. Voloshin and L. B. Okun, JETP Lett. **23**, 333 (1976) [Pisma Zh. Eksp. Teor. Fiz. **23**, 369 (1976)];
 - [11] N. A. Tornqvist, Phys. Rev. Lett. **67**, 556 (1991); N. A. Tornqvist, Z. Phys. C **61**, 525 (1994) [hep-ph/9310247]; N. A. Tornqvist, Phys. Lett. B **590**, 209 (2004) [hep-ph/0402237].
 - [12] M. Karliner and H. J. Lipkin, arXiv:0802.0649 [hep-ph]; M. Karliner and J. L. Rosner, Phys. Rev. Lett. **115** (2015) no.12, 122001 [arXiv:1506.06386 [hep-ph]]; M. Karliner, Acta Phys. Polon. B **47**, 117 (2016).
 - [13] C. E. Thomas and F. E. Close, Phys. Rev. D **78**, 034007 (2008) [arXiv:0805.3653 [hep-ph]]; F. Close, C. Downum and C. E. Thomas, Phys. Rev. D **81**, 074033 (2010) [arXiv:1001.2553 [hep-ph]].
 - [14] S. Ohkoda, Y. Yamaguchi, S. Yasui, K. Sudoh and A. Hosaka, Phys. Rev. D **86**, 034019 (2012) [arXiv:1202.0760 [hep-ph]]; S. Ohkoda, Y. Yamaguchi, S. Yasui, K. Sudoh and A. Hosaka, arXiv:1209.0144 [hep-ph].
 - [15] M. T. AlFiky, F. Gabbiani and A. A. Petrov, Phys. Lett. B **640**, 238 (2006) [hep-ph/0506141]; I. W. Lee, A. Faessler, T. Gutsche and V. E. Lyubovitskij, Phys. Rev. D **80**, 094005 (2009) [arXiv:0910.1009 [hep-ph]]; M. Suzuki, Phys. Rev. D **72**, 114013 (2005) [hep-ph/0508258]; J. R. Zhang, M. Zhong and M. Q. Huang, Phys. Lett. B **704**, 312 (2011) [arXiv:1105.5472 [hep-ph]]; D. V. Bugg, Europhys. Lett. **96**, 11002 (2011) [arXiv:1105.5492 [hep-ph]]; J. Nieves and M. P. Valderrama, Phys. Rev. D **84**, 056015 (2011) [arXiv:1106.0600 [hep-ph]]; M. Cleven, F. K. Guo, C. Hanhart and U. G. Meissner, Eur. Phys. J. A **47**, 120 (2011) [arXiv:1107.0254 [hep-ph]]; T. Mehen and J. W. Powell, Phys. Rev. D **84**, 114013 (2011) [arXiv:1109.3479 [hep-ph]]; F. K. Guo, C. Hidalgo-Duque, J. Nieves and M. P. Valderrama, Phys. Rev. D **88**, 054007 (2013) [arXiv:1303.6608 [hep-ph]]; Q. Wang, C. Hanhart and Q. Zhao, Phys. Rev. Lett. **111**, no. 13, 132003 (2013) [arXiv:1303.6355 [hep-ph]]; F. K. Guo, C. Hanhart, Q. Wang and Q. Zhao, Phys. Rev. D **91** (2015) no.5, 051504 [arXiv:1411.5584 [hep-ph]]; X. W. Kang, Z. H. Guo and J. A. Oller, Phys. Rev. D **94** (2016) no.1, 014012 [arXiv:1603.05546 [hep-ph]]; M. Al-

- baladejo, F. K. Guo, C. Hidalgo-Duque and J. Nieves, Phys. Lett. B **755**, 337 (2016) [arXiv:1512.03638 [hep-ph]].
- [16] E. S. Swanson, Phys. Rept. **429**, 243 (2006) [hep-ph/0601110]; Z. F. Sun, J. He, X. Liu, Z. G. Luo and S. L. Zhu, Phys. Rev. D **84**, 054002 (2011) [arXiv:1106.2968 [hep-ph]].
- [17] Y. Liu and I. Zahed, Phys. Lett. B **762**, 362 (2016) [arXiv:1608.06535 [hep-ph]]; Y. Liu and I. Zahed, arXiv:1610.06543 [hep-ph].
- [18] A. V. Manohar and M. B. Wise, Nucl. Phys. B **399**, 17 (1993) [hep-ph/9212236]; N. Brambilla *et al.*, Eur. Phys. J. C **71**, 1534 (2011) [arXiv:1010.5827 [hep-ph]]; M. B. Voloshin, Prog. Part. Nucl. Phys. **61**, 455 (2008) [arXiv:0711.4556 [hep-ph]]; J. M. Richard, arXiv:1606.08593 [hep-ph].
- [19] D. O. Riska and N. N. Scoccola, Phys. Lett. B **299**, 338 (1993).
- [20] M. A. Nowak, I. Zahed and M. Rho, Phys. Lett. B **303**, 130 (1993).
- [21] S. Chernyshev, M. A. Nowak and I. Zahed, Phys. Rev. D **53**, 5176 (1996) [hep-ph/9510326].
- [22] M. Nielsen, F. S. Navarra and S. H. Lee, Phys. Rept. **497**, 41 (2010) [arXiv:0911.1958 [hep-ph]].
- [23] A. Paredes and P. Talavera, Nucl. Phys. B **713**, 438 (2005) [hep-th/0412260]; J. Erdmenger, N. Evans and J. Grosse, JHEP **0701**, 098 (2007); [hep-th/0605241]. J. Erdmenger, K. Ghoroku and I. Kirsch, JHEP **0709** (2007) 111 [arXiv:0706.3978 [hep-th]]; C. P. Herzog, S. A. Stricker and A. Vuorinen, JHEP **0805**, 070 (2008) [arXiv:0802.2956 [hep-th]]; Y. Bai and H. C. Cheng, JHEP **1308**, 074 (2013) [arXiv:1306.2944 [hep-ph]]; K. Hashimoto, N. Ogawa and Y. Yamaguchi, JHEP **1506**, 040 (2015) [arXiv:1412.5590 [hep-th]]. J. Sonnenschein and D. Weissman, arXiv:1606.02732 [hep-ph].
- [24] Y. Liu and I. Zahed, arXiv:1611.03757 [hep-ph].
- [25] T. Sakai and S. Sugimoto, Prog. Theor. Phys. **113**, 843 (2005) [hep-th/0412141]; T. Sakai and S. Sugimoto, Prog. Theor. Phys. **114**, 1083 (2005) [hep-th/0507073].
- [26] T. Fujiwara, T. Kugo, H. Terao, S. Uehara and K. Yamawaki, Prog. Theor. Phys. **73**, 926 (1985).
- [27] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [hep-th/9711200]; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [hep-th/9802109]; E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998) [hep-th/9803131]; I. R. Klebanov and E. Witten, Nucl. Phys. B **556**, 89 (1999) [hep-th/9905104].
- [28] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **95**, 261602 (2005) [hep-ph/0501128]; L. Da Rold and A. Pomarol, Nucl. Phys. B **721**, 79 (2005) [hep-ph/0501218].
- [29] S. Hong, S. Yoon and M. J. Strassler, JHEP **0604**, 003 (2006) [hep-th/0409118]; J. Erlich, G. D. Kribs and I. Low, Phys. Rev. D **73**, 096001 (2006) doi:10.1103/PhysRevD.73.096001 [hep-th/0602110]; H. R. Grigoryan and A. V. Radyushkin, Phys. Rev. D **76**, 095007 (2007) [arXiv:0706.1543 [hep-ph]]; H. R. Grigoryan and A. V. Radyushkin, Phys. Lett. B **650**, 421 (2007) [hep-ph/0703069]; S. S. Afonin and I. V. Pusekov, EPJ Web Conf. **125**, 04004 (2016) [arXiv:1606.06091 [hep-ph]]; N. R. F. Braga, M. A. Martin Contreras and S. Diles, Europhys. Lett. **115**, no. 3, 31002 (2016) [arXiv:1511.06373 [hep-th]]; A. Gorsky, S. B. Gudnason and A. Krikun, Phys. Rev. D **91**, no. 12, 126008 (2015) [arXiv:1503.04820 [hep-th]].
- [30] G. F. de Teramond, S. J. Brodsky, A. Deur, H. G. Dosch and R. S. Sufian, arXiv:1611.03763 [hep-ph]; H. G. Dosch, G. F. de Teramond and S. J. Brodsky, Phys. Rev. D **92** (2015) no.7, 074010 [arXiv:1504.05112 [hep-ph]].
- [31] <http://mathworld.wolfram.com/>; dlmf.nist.gov/10.
- [32] A. Aoki *et al.*, Phys. Rev. Lett. **80**, 5711 (1998).
- [33] A. F. Falk and M. E. Luke, Phys. Lett. B **292**, 119 (1992) [hep-ph/9206241].
- [34] A. Karch, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. D **74**, 015005 (2006) [hep-ph/0602229].
- [35] U. Gursoy and E. Kiritsis, JHEP **0802**, 032 (2008) [arXiv:0707.1324 [hep-th]]; U. Gursoy, E. Kiritsis and F. Nitti, JHEP **0802**, 019 (2008) [arXiv:0707.1349 [hep-th]].